Institute of Mathematics and Informatics
Bulgarian Academy of Sciences

International Workshop
"Hausdorff Geometry of Polynomials"

in honour of Academician Blagovest Sendov

July 10-15, 2023, Sofia, Bulgaria

in the frame of the

Fourth International Conference

MATHEMATICS DAYS IN SOFIA

https://mds.math.bas.bg/w-hgp/

Organizers

Institute of Mathematics and Informatics, Bulgarian Academy of Sciences
Union of the Bulgarian Mathematicians
Faculty of Mathematics and Informatics, Sofia University "St. Kliment Ohridski"

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Aim and Scope

The 'Hausdorff Geometry of Polynomials' workshop was inspired by Academician Blagovest Sendov's fundamental contributions to the study of equilibrium points for polynomials. The workshop provides an opportunity for experts on the subject to exchange latest results dealing with polynomial functions and related topics such as optimal prediction methods, electrostatics, and minimum energy problems. Furthermore, the workshop provides an opportunity for these distinguished researchers to combine forces to tackle challenging open problems in the spirit of Sendov's famous conjecture.

Organising Committee

Edward Saff (Vanderbilt University, USA)
Boris Shapiro (Stockholm University, Sweden)
Andrei Martinez-Finkelshtein (Baylor University, USA)
Peter Boyvalenkov (Institute of Mathematics and Informatics, Bulgaria)

Participants

Andrei Martinez-Finkelstein, Baylor University, USA
Arno Kuijlaars, KU Leuven, Belgium
Boris Shapiro, Stockholm University, Sweden
Dmitrii Karp, Holon Institute of Technology, Israel
Edward Saff, Vanderbilt University, USA
Franck Wielonsky, Université Aix-Marseille, France
Hristo Sendov, University of Western Ontario, Canada
Liudmyla Kryvonos, Vanderbilt University, USA
Nikos Stylianopoulos, University of Cyprus, Cyprus
Peter Boyvalenkov, Institute of Mathematics and Informatics, Bulgaria
Ramon Orive, Universidad de la Laguna, Spain
Vladimir Kostov, Université Côte dAzur, France

Venue

The workshop will be held at Grand Hotel Sofia (Sofia, 1 Gurko str.), Triadica hall.

Program Schedule

Monday, July 10

14:00-14:50 Hristo Sendov, *Stronger Rolle's theorem for complex polynomials*

15:30-16:20 Arno Kuijlaars, *Matrix orthogonal polynomials and random tilings*

16:30-17:20 Dmitrii Karp, *Polynomials arising in hyper-geometric research: motivation, properties and conjectures*

Tuesday, July 11

14:00-14:50 Andrei Martinez-Finkelshtein, *On the flow of zeros of derivatives of polynomials*

15:30-16:20 Vladimir Kostov, *Beyond Descartes' rule of signs*

16:30-17:20 Nikos Stylianopoulos, *Strong comparison law for Faber and Bergman*

Wednesday, July 12

14:00-14:50 Franck Wielonsky, *Optimal prediction measures and polynomials of extremal growth*

15:30-16:20 Lyudmila Kryvonos, *A constrained logarithmic energy problem on the unit circle*

16:30-17:30 Open Problems

Thursday, July 13

14:00-14:50 Boris Shapiro, *Introducing isodynamic points for binary forms and their ratios*

15:30-16:20 Edward Saff, *Threshold condensation to singular support for a Riesz equilibrium problem*

16:30-17:30 Open problems

Friday, July 14

14:00-14:50 Ramon Orive, *Electrostatic partners or… How can we find an electrostatic interpretation for the zeros of multiple orthogonal polynomials?*

15:30-16:20 Open Problems

16:30-17:20 Open Problems

Abstracts

Andrei Martinez-Finkelstein

On the flow of zeros of derivatives of polynomials

Assume we have a sequence of polynomials whose asymptotic zero distribution is known. What can be said about the zeros of their derivatives? Especially if we differentiate each polynomial several times, proportional to their degree?

This simple-to-formulate problem has recently attracted the attention of several researchers. Both the problem and the methods of its solution have exciting connections with free probability, random matrices, and approximation theory on the complex plane. This is a partial survey in which I will explain some known results in this direction and our approach to the problem. This is a joint work in progress with E. Rakhmanov from the University of South Florida.

Arno Kuijlaars

Matrix orthogonal polynomials and random tilings

The talk is about certain matrix polynomials with non-hermitian orthogonality on a contour in the complex plane. These matrix orthogonal polynomials arise in the analysis of random tilings of planar domains with periodic weightings. I will focus on a particular case of a three-periodic lozenge tiling of a hexagon. The matrix orthogonality is used to obtain the Arctic curves that separate the asymptotic phases of the model, known as the frozen, smooth and rough phases.

Boris Shapiro

Introducing isodynamic points for binary forms and their ratios

The isodynamic points of a plane triangle are known to be the only pair of its centers invariant under the action of the Möbius group M on the set of triangles. Generalizing this classical result, we introduce below the isodynamic map associating to a univariate polynomial of degree $d \geq 3$ with at most double roots a polynomial of degree (at most) $2d - 4$ such that this map commutes with the action of the Möbius group M on the zero loci of the initial polynomial and its image. The roots of the image polynomial will be called the isodynamic points of the preimage polynomial. Our construction naturally extends from univariate polynomials to binary forms and further to their ratios. Joint work with Ch. Hägg and M. Shapiro.

Dmitrii Karp

Polynomials arising in hypergeometric research: motivation, properties and conjectures

In the talk, we will discuss three hypergeometric-related themes centered around certain polynomials. Our first theme is concerned with d-dimensional (or generalized)

Narayana polynomials, whose coefficients count several important combinatorial objects. Starting with the hypergeometric generating function of these polynomials, we show how certain relatively recent hypergeometric transformations lead to many known and even some new properties of these polynomials. In our second theme, we deal with a scalar Riemann-Hilbert problem solved by the ratio of the Gauss hypergeometric functions with shifted parameters. The key factor of the boundary data turns out to be a special polynomial whose properties define whether a given ratio belongs to a generalized Nevanlinna class (in particular, if it is a Markov function). We reveal the connection of this polynomial to recently discovered duality relations for the generalized hypergeometric functions. Our third theme is motivated by a log-concavity problem for generic series containing rising factorials. This problem leads naturally to a generic class of polynomials expressed in rising factorial basis. We will establish positivity of their coefficients in monomial basis and formulate a number of conjectures, some of which can be interpreted as nonlinear transformations preserving finite PF_∞ sequences.

Edward Saff

Threshold condensation to singular support for a Riesz equilibrium problem

We compute the equilibrium measure in dimension $d=s+4$ associated to a Riesz s-kernel interaction with an external field given by a power of the Euclidean norm. Our study reveals that the equilibrium measure can be a mixture of a continuous part and a singular part. Depending on the value of the power, a threshold phenomenon occurs and consists of a dimension reduction or condensation on the singular part. In particular, in the logarithmic case $s=0$ ($d=4$), there is condensation on a sphere of special radius when the power of the external field becomes quadratic. This contrasts with the case $d=s+3$ studied previously, which showed that the equilibrium measure is fully dimensional and supported on a ball. Our approach makes use, among other tools, of the Frostman or Euler-Lagrange variational characterization, the Funk-Hecke formula, the Gegenbauer orthogonal polynomials, and hypergeometric special functions. Joint work with Djalil Chafaï and Robert S. Womersley.

Franck Wielonsky

Optimal prediction measures and polynomials of extremal growth

We review some properties of optimal prediction measures (OPM), a notion related to optimal designs in statistics. The study of OPM's is connected with many classical notions in approximation and potential theory, such as the Bergman kernel, the Christoffel function, polynomials of extremal growth, Faber polynomials, the Szego function and balayage of measures. We describe some of these connections and give some hints on the asymptotic behavior of OPM's. Joint work with Stephane Charpentier (Marseille) and Norm Levenberg (Bloomington).

Hristo Sendov

Stronger Rolle's theorem for complex polynomials

Every Calculus student is familiar with the classical Rolle's theorem stating that if a real polynomial *p* satisfies *p*(-1) = *p*(1), then it has a critical point in (-1,1). In 1934, L. Tschakaloff strengthened this result by finding a *minimal* interval, contained in (-1,1), that holds a critical point of every real polynomial with *p*(-1)=*p*(1), up to a fixed degree. In 1936, he expressed a desire to find an analogue of his result for complex polynomials.

This talk will present the main stepping stones on the road to the following strengthening of Rolle's theorem for complex polynomials. If *p*(*z*) is a complex polynomial of degree *n*≥5, satisfying *p*(-*i*)=*p*(*i*), then there is at least one critical point of *p* in the union *D*[-*c*;*r*] ∪ *D*[*c*;*r*] of two closed disks with centres -*c*, *c* and radius *r*, where *c*= cot (2π/*n*), *r*=1/sin (2π/*n*). If *n*=3, then the closed disk *D*[0; $1/\sqrt{3}$] has this property; and if *n*=4 then the union of the closed disks *D*[-1/3; 2/3] ∪ *D*[1/3; 2/3] has this property. In the last two cases, the domains are minimal, with respect to inclusion, having this property. This theorem is stronger than any other known Rolle's Theorem for complex polynomials of any degree. A minimal Rolle's domains are found for polynomials of degree 3 and 4, answering Tschakaloff's question. This is a joint work with Blagovest Sendov from the Bulgarian Academy of Sciences.

Liudmyla Kryvonos

A constrained logarithmic energy problem on the unit circle

We study the problem of minimizing the logarithmic energy, $\mathcal{E}(\mu) \coloneqq \int\int log\frac{1}{|z-\zeta|}d\mu(z)d\mu(\zeta)$, of probability measures μ supported on the unit circle with an additional constraint imposed on the mass of a fixed subarc. Namely, for given θ, $0 < \theta < 2\pi$, and given q, $0 < q < 1$, we determine the measure μ^*, such that $\mathcal{E}(\mu^*) \coloneqq inf\{\mathcal{E}(\mu): \mu \in \mathcal{P}(\mathbb{S}^1), \mu(A_\theta) = q\}$, where A_θ is the arc from $e^{-i\theta/2}$ to $e^{i\theta/2}$. The result answers a question raised by E. Meckes in connection with the characterization of the behavior of eigenvalues for random unitary matrices. Joint work with E. Saff.

Nikos Stylianopoulos

Strong comparison law for Faber and Bergman

Let G be a bounded simply-connected domain in the complex plane $\mathbb{C}$, with boundary Γ. The Bergman polynomials $\{p_n(z)\}_{n=0}^{\infty}$ of G are defined by

$$\int p_n(z)\overline{p_m(z)}\, dA(z) = \delta_{m,n}, p_n(z) = \lambda_n z^n + \cdots, \lambda_n > 0,$$

where $dA(z)$ denotes the area measure on G.

Let Φ be the conformal map $\overline{\mathbb{C}}\backslash\overline{G} \to \{w: |w| > 1\}$, such that

$$\Phi(z) = \gamma z + \gamma_0 + \gamma_1\frac{1}{z} + \cdots, \gamma > 0.$$

Then, the normalised Faber polynomials $\{f_n(z)\}_{n=0}^{\infty}$ for G are defined as the polynomial parts of $\Phi'(z)\Phi^n(z)$, normalised so that

$$f_n(z) = \sqrt{\frac{n+1}{\pi}}\gamma^{n+1}z^n + \cdots .$$

Under the assumption that Γ is a quasi-conformal curve we establish the (sharp in both ends) comparison law

$$(1 - \|C\|^2)\sum_{j=0}^{n}\left|p_j(z)\right|^2 \leq \sum_{j=0}^{n}\left|f_j(z)\right|^2 \leq \sum_{j=0}^{n}\left|p_j(z)\right|^2, z \in \mathbb{C},$$

where $\|\cdot\|$ denotes the 2-norm of a bounded linear operator in l_2 and C (with $\|C\| < 1$) is the Grunsky matrix associated with Γ.
In this talk we show how the comparison law can be applied to yield asymptotics for the Bergman, Faber and the Christoffel functions $\lambda_n(z) := \frac{1}{\sum_{j=0}^{n}\left|p_j(z)\right|^2}$, of G.

Ramon Orive

Electrostatic partners or... How can we find an electrostatic interpretation for the zeros of multiple orthogonal polynomials?

T. J. Stieltjes (1856-1894) found an elegant way to describe the electrostatic interpretation for the zeros of classical orthogonal polynomials (Jacobi, Laguerre, Hermite), which he himself extended to the case of the currently called Heine-Stieltjes polynomials. In this talk we discuss the extension of this approach to the case of multiple (or Hermite-Padé) orthogonal polynomials, especially focusing on the most studied cases in the literature, i.e. the Angelsco's and Nikishin's settings.
Joint work with A. Martínez Finkelshtein and J. Sánchez Lara.

Vladimir Kostov

Beyond Descartes' rule of signs

A univariate real degree *d* polynomial is hyperbolic if all its roots are real. By Descartes' rule of signs, such a polynomial with non-vanishing coefficients has $\tilde{c}$ positive and $\tilde{p}$ negative roots, $\tilde{c} + \tilde{p} = d$, where $\tilde{c}$ and $\tilde{p}$ denote the numbers of sign changes and sign preservations in the sequence of coefficients. We ask the questions:
1. When the moduli of the roots (supposed all distinct) are ordered on the real positive half-axis, which positions can the moduli of the negative roots occupy?
2. When the positions of the sign changes and preservations are known, what the order of the moduli of roots can be?
In the talk we say in which cases the order of moduli of roots determines the positions of the sign changes or vice versa, and we present further results related to the above two questions.