# The squeezing function: exact computations and a new application

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(joint work with **Diganta Borah and Sushil Gorai**)

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- $\exists F_z$ s.t. $s_D(z;F_z) = s_D(z)$.
- (From above fact) If $\exists z \in D$ s.t. $s_D(z) = 1$, then $D \cong B^n(0,1)$.

## Holomorphic homogeneous regular domains

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- (Borah, Gorai & B.) Starlike weakly linearly convex domains.

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That said, it is not easy to determine when a domain $D \Subset \mathbb{C}^n$ is HHR since, to do so, one would need information on $s_D(z)$ for **every** $z \in D \ldots$

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Theorem 2. *Let $D$ be a bounded symmetric domain. Then, for any $z \in D$, $s_D(z) = 1/\sqrt{\mathrm{rank}(D)}$.*

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*Proof:* Let $F : D \to B^n(0;1)$ be injective, holomorphic, and s.t. $F(z) = 0$. Let $B^n(0, R) \subset F(D)$. Write $(g_1, \dots, g_n) := F \circ \varphi$ and

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- As each $g_i \in H^2(\mathbb{D}^p)$

$$\sum_{i=1}^n \Big[ \sum\nolimits_{\alpha\in\mathbb{N}^p} |C_\alpha^{(i)}|^2 \Big] = \lim_{r\to 1^-} \sum_{i=1}^n \frac{1}{(2\pi)^p} \int_{[0,2\pi]^p} |g_i(re^{i\theta_1}, \dots re^{i\theta_p})|^2 \, d\Theta \leq 1.$$

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- Fatou's Theorem applied to $g_{i,\,j} := g_i(\bullet\boldsymbol{\epsilon}_j)$ $(j = 1, \dots, p)$ assures existence of the boundary-values map $g^{\bullet}_{i,\,j}(e^{i\tau})$ for a.e. $\tau \in [0, 2\pi]$.

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- From the two bullet-points – summing over $j$ in (1) – we have

$$pR^2 \;\leq\; 1$$
$$\Rightarrow s_D(z; F)^2 \;\leq\; 1/p.$$

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- Let $\varphi : \mathbb{D}^p \to D$ be given by $\varphi := (\varphi_1, \dots, \varphi_p)$. By the last result, $s_D(z) \leq 1/\sqrt{p} \leq 1/\sqrt{m}$. ■

# The proof of Theorem 2: set-up

Our proof, in contrast to **other** previous partial results avoids treating bounded symmetric domains case-by-case.

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PROPOSITION. *Let $D$ be a realisation of an irreducible bounded symmetric domain as a bounded convex balanced domain in $\mathbb{C}^n$. Let $r$ be the rank of $D$ and let $K := \{g \in \mathrm{Aut}(D) : g(0) = 0\}$. Fix a $K$-invariant inner product $(\cdot\,|\,\cdot)$ and write $\|x\|_D := (x\,|\,x)^{1/2}$.*

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The above result follows from the study of a non-associative algebra called a *Hermitian Jordan triple system* associated with a Harish-Chandra realisation of $D$.

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Clearly, $s_D \geq 1/\sqrt{r}$. So, when $D$ is irreducible, we are done. "■"