# Aspects of B-regularity

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**Definition**

A compact set $K \subseteq \mathbb{C}^n$ is called **B-regular** (short for Bremermann regular) if every continuous function on $K$ can be approximated uniformly on $K$ by functions which are continuous and plurisubharmonic on some (varying) neighborhoods of $K$.

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- for each $z \in K$ the Dirac delta $\delta_z$ is the only probability measure $\mu$ (called Jensen measure) such that

$$u(z) \leqslant \int_K u d\mu$$

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- For each $M > 0$ there exists a smooth ($C^\infty$) function $u$ which is plurisubharmonic on some neighborhood of $K$ such that $0 < u < 1$ and $dd^c u \geqslant Mdd^c\|z\|^2$ (Sibony'87).

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- Every upper semicontinuous function on $K$ can be approximated by a sequence of continuous plurisubharmonic functions defined on (varying) neighborhoods of $K$ that is decreasing on $K$. (Poletsky'96, Czyż-Hed-Persson'12, D-D direct proof)

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- Totally disconnected sets. This is because they are holomorphically convex in the sense presented in the book of Stout, so every complex valued continuous function on such a set can be uniformly approximated by holomorphic functions defined on varying neighborhoods.

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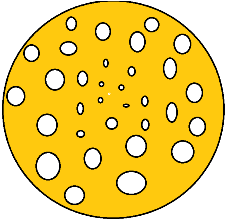
- More generally, any compact set containing an analytic disc. (In particular, 2 is the threshold of Hausdorff dimension up to which vanishing measure implies B-regularity.)
- Some "fat enough" Swiss-cheese type compact sets in the complex plane.



We collect some of the properties of B-regular compact sets:

- If $K_j$ are compact B-regular and $\bigcup_{j=1}^{\infty} K_j = K$ is compact then $K$ is B-regular.

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- The intersection of a B-regular compact set with lower dimensional affine complex subspace is again B-regular in the corresponding subspace.
- B-regularity can not be explained by sections (slices). For example, the analytic disk $\overline{\mathbb{D}(0,1)} \ni z \to (z, z^2) \in \mathbb{C}^2$ is not B-regular, whereas the intersection with any complex line has at most two points, and hence is B-regular.

# Behavior under mappings

- Very simple transformations can destroy the B-regularity. For example $\mathbb{C}^2 \ni (z, w) \to z + w \in \mathbb{C}$ takes the compact B-regular $(\mathcal{C} + i\mathcal{C}) \times (\mathcal{C} + i\mathcal{C})$ to $[0, 2] + i[0, 2]$, where $\mathcal{C}$ is the ternary Cantor set.

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- If $X \subseteq \mathbb{C}^n$ and $Y \subseteq \mathbb{C}^m$ are compact subsets, and $\varphi : X \to Y$ is a continuous mapping such that all its components are uniformly approximable on $X$ by holomorphic functions on neighborhoods. If $Y$ is B-regular and for any $y \in Y$ one has that $\varphi^{-1}(y)$ is B-regular then $X$ is B-regular.

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- Let $X \subseteq \mathbb{C}^n$ and $Y \subseteq \mathbb{C}^m$ be compact subsets, and $\varphi : X \to Y$ be a continuous mapping that extends to a proper holomorphic mapping between neighborhoods of $X$ and $Y$, $X \subseteq \Omega$, $Y \subseteq U$. If $X$ is B-regular then $Y$ is B-regular.

We can not relax the condition that $\varphi$ has a holomorphic extension to a proper mapping between neighborhoods by assuming that $\varphi$ is a mapping that can be approximated even by holomorphic automorphisms uniformly on $X$ as the mapping

$$\mathbb{C}^2 \ni (z, w) \to (z + w, 0) \in \mathbb{C}^2$$

can be uniformly approximated on $\overline{\mathbb{D}(0,2)} \times \overline{\mathbb{D}(0,2)} \supseteq (\mathcal{C} + i\mathcal{C}) \times (\mathcal{C} + i\mathcal{C})$ by the automorphisms

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A characterization of B-regularity in $\mathbb{C}^1$ is possible with the use of the so-called **fine topology**:

A compact $K \subseteq \mathbb{C}^1$ is B-regular if and only if $K$ has empty interior in the fine topology, established in Sibony'87.

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Take a Swiss cheese

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- The disks and the complement of $\mathbb{D}(0,1)$ have pairwise disjoint closures. This yields the restriction $\sum_{j,k=1}^{\infty} r_{j,k}^2 < 1$.
- The disks are "evenly distributed" so that $K$ has empty interior.
- The disks with fixed $j$ are situated within a specified annulus centered at 0, $\overline{\mathbb{D}(z_{j,k}, r_{j,k})} \subseteq \left\{ z \in \mathbb{C} : \frac{1}{2^j} < |z| < \frac{1}{2^{j-1}} \right\}$. This yields the restriction $\sum_{k=1}^{\infty} r_{j,k}^2 < \frac{1}{2^{2(j-1)}} - \frac{1}{2^{2j}}$.
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Choosing $r_{j,k} = \frac{1}{2^{2^k (j+1)^3 - 1}}$ is good (The idea is to make 0 a thin boundary point of $\mathbb{D}(0,1) \setminus K$, so it belongs to the fine interior of $K$)

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D-D: no with example (complicated, uses finely holomorphic functions that can not be extended to holomorphic ones).

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D-D: It is not true that any continuous function on a B-regular compact set $K$ can be approximated uniformly on $K$ by functions pluriharmonic on (varying) neighborhoods of $K$.

Sketch Consider the set $K = \partial B(0,1) \cup L$ where $L$ is the closed linear segment joining the origin with the point $(1,0,\ldots,0) \in \partial B(0,1)$. The set $K$ is clearly B-regular, yet the function $u(z) = 1 - \|z\|$ can not be approximated uniformly on $K$ by pluriharmonic functions, since any pluriharmonic function on a neighborhood of $K$ would extend to a pluriharmonic function on the ball $B(0,1)$. By uniqueness, the extension is close to 1 near the origin and close to 0 near $\partial B(0,1)$. This this violates the maximum principle.

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- the sets $\Omega_k$ where the approximating functions $u_k$ are plurisubharmonic can vary in such a way that $\bigcap_{k=1}^{\infty} \Omega_k = K$, and hence there is no "domain" on which the approximating sequence can converge to something plurisubharmonic.

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- even if $K \subseteq \Omega = int \bigcap_{k=1}^{\infty} \Omega_k$ then it may happen that $u_k$ converges to $u$ only on $K$ and diverges elsewhere on $\Omega$. For example the sequence $u_k(z) = k \ln |z + \sqrt{z^2 - 1}| - \frac{1}{k}$ consists of global subharmonic functions but converges uniformly to $u = 0$ only on the interval $[-1, 1]$ and diverges elsewhere.

Let $X$ be the set $[-1,1]\times\{0\}\cup\{0\}\times[-1,1]$.

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Whatever the proposed subharmonic extension of $-|z|^2$, which we call $u$, to some $\Omega \supseteq X$ is its partial derivatives depend only on the values on the axes, so $u_x\big|_{(-1,1)\times 0} = (-x^2 - y^2)\big|_{(-1,1)\times 0} = -2x, u_y\big|_{\{0\}\times(-1,1)} = -2y$ and hence $u_{xx}(0,0) + u_{yy}(0,0) = -4$ which is a contradiction.

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$$z \to \frac{z}{\sqrt[n]{4}}\left(1 + \frac{1}{z^n}\right)^{\frac{2}{n}} = \frac{z}{\sqrt{2}}\left(1 + \frac{1}{z^4}\right)^{\frac{1}{2}},$$

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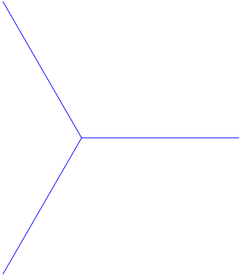
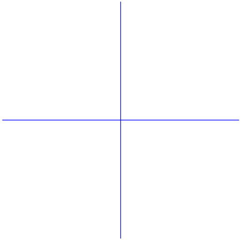
On the other hand, we have D-D: If $-\|z\|^2$ can be extended then any $C^{1,1}$ function on $X$ (a trace of a $C^{1,1}$ function on a neighborhood) can be extended.





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