The Julia-Wolff-Carathéodory Theorem in convex domains of finite type

(joint work with L. Arosio)

Matteo Fiacchi

University of Ljubljana

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University of Ljubljana

Fatou's theorem

Theorem (Fatou's)

$f : \mathbb{D} \longrightarrow \mathbb{D}$ holomorphic. Then

$$f(\xi) := \lim_{t\to 1} f(t\xi)$$

exists a.e. in $\partial\mathbb{D}$.

Definition (Non-tangential limit)

$f : \mathbb{D} \to \mathbb{C}$ a function, $\xi \in \partial\mathbb{D}$,

$$\angle \lim_{z\to\xi} f(z) = L$$

if for all sequence $(z_k)_k$ converging to ξ non-tangentially (i.e. inside a cone) we have $f(z_k) \to L$.

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Dilation and Julia's Lemma

Definition

$f : \mathbb{D} \to \mathbb{D}$ holomorphic, $\xi \in \partial\mathbb{D}$. The **dilation** λ_ξ at ξ is defined by

$$\lambda_\xi := \liminf_{z \to \xi} \frac{1 - |f(z)|}{1 - |z|} \in (0, +\infty].$$

The point ξ is **regular contact point** if $\lambda_\xi < +\infty$.

Theorem (Julia's Lemma)

$f : \mathbb{D} \longrightarrow \mathbb{D}$ holomorphic and $\xi \in \partial\mathbb{D}$ regular contact, then $\exists!\ \eta \in \partial\mathbb{D}$ such that

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Remark

$$\log \lambda_\xi = \liminf_{z\to\xi} k_{\mathbb{D}}(z, 0) - k_{\mathbb{D}}(f(z), 0).$$

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The unit ball $\mathbb{B}^d$

Definition (Koranyi region)

The Koranyi region of amplitude $M > 1$ and vertex $\xi \in \partial\mathbb{B}^d$ is

$$K(\xi, M) = \left\{ z \in \mathbb{B}^d : \frac{|1 - \langle z, \xi \rangle|}{1 - ||z||} < M \right\}.$$

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($q = 2$ for clarity) $f : \mathbb{B}^2 \longrightarrow \mathbb{B}^2$ holomorphic and $\xi \in \partial\mathbb{B}^2$ regular contact. *Write $d_z f$ as*

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The Restricted K-limit

Definition (Special restricted)

A sequence $(z_k) \to \xi$ in $\mathbb{B}^d$ is **special restricted** if z_k is contained in a Korainy region and

$$\frac{||z_k - \langle z_k, \xi\rangle \xi||^2}{1 - |\langle z_k, \xi\rangle|^2} \to 0$$

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$f : \mathbb{B}^d \to \mathbb{C}$ a function, $\xi \in \partial\mathbb{B}^d$,

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In strongly convex domains

Definition

$D \subset \mathbb{C}^d$ strongly convex, $f : D \to D$, $\xi \in \partial D$, $p \in D$

$$\log \lambda_{\xi,p} = \liminf_{z \to \xi} k_D(z,p) - k_D(f(z),p);$$

$$K_p(\xi, M) = \left\{ z \in D : k_D(z,p) + \lim_{w \to \xi} k_D(z,w) - k_D(w,p) < 2 \log M \right\}.$$

The existence of the limit

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was proved by Abate (existence of horospheres!).

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$\phi : \mathbb{D} \to D$ holomorphic is a **complex geodesic** if for all $z, w \in \mathbb{D}$

$$k_D(\phi(z), \phi(w)) = k_{\mathbb{D}}(z, w).$$

If D is a bounded convex domain, then there exists $\tilde{\rho} : D \to \mathbb{D}$ holomorphic **left inverse** such that

$$\tilde{\rho} \circ \phi = \mathrm{Id}_{\mathbb{D}}.$$

Notice that $\rho := \phi \circ \tilde{\rho}$ is a holomorphic retraction (i.e. $\rho^2 = \rho$).

Theorem

If $D \subset \mathbb{C}^d$ is a strongly convex domain, then for all $p \in D$ and $\xi \in \partial D$ there exists an unique complex geodesic ϕ with $\phi(0) = p$ and $\phi(1) = \xi$. Moreover, there exists $\phi'(1)$ and is transversal to $T^{\mathbb{C}}_{\xi}(\partial D)$.

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K'-limit in strongly convex domains

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Fix ϕ complex geodesic with $\phi(1) = \xi$. A sequence $(z_k) \to \xi$ in D is **special restricted** if z_k is contained in a Korainy region and

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Theorem (Abate's JWC)

$D \subset \mathbb{C}^d$ strongly convex, $f : D \longrightarrow D$ holomorphic, $\xi \in \partial D$ regular contact point, $p \in D$. Let ϕ and ψ be the (unique) complex geodesics with $\phi(0) = \psi(0) = p$, $\phi(1) = \xi$ and $\psi(1) = \eta$. Then

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Complex geodesics in convex finite type domains

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The derivative $\phi'(1)$ may not exists and the approach can be tangential.

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We don't know if the complex geodesics are unique.

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Kobayashi type

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D convex domain, $\xi \in \partial D$ smooth point and $v \in \mathbb{C}^d$, $v \neq O$. The **Kobayashi type** at ξ in the direction v is the number

$$s_\xi(v) = \inf \{ s : d(z, \partial D)^s \kappa_D(z, v) \text{ is bounded on Koranyi regions} \}$$

Proposition (Abate-Tauraso)

- *if* $v \notin T_\xi^{\mathbb{C}}(\partial D)$ *then* $s_\xi(v) = 1$;
- *if* $v \in T_\xi^{\mathbb{C}}(\partial D)$ *then* $1/type(\xi) \leq s_\xi(v) \leq 1 - 1/type(\xi)$.

Question (Abate)

Is $1/s_\xi(v)$ *equal to* $L_\xi(v)$*, the line type at* ξ *in the direction* v*?*

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- *if* $v \notin T_\xi^{\mathbb{C}}(\partial D)$ *then* $s_\xi(v) = 1$;
- *if* $v \in T_\xi^{\mathbb{C}}(\partial D)$ *then* $1/type(\xi) \leq s_\xi(v) \leq 1 - 1/type(\xi)$.

Question (Abate)

Is $1/s_\xi(v)$ *equal to* $L_\xi(v)$*, the line type at* ξ *in the direction* v*?*

Kobayashi type

Definition (Kobayashi type)

D convex domain, $\xi \in \partial D$ smooth point and $v \in \mathbb{C}^d$, $v \neq O$. The **Kobayashi type** at ξ in the direction v is the number

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Theorem (Abate-Tauraso's JWC)

$D \subset \mathbb{C}^d$ convex finite type + [technical assumptions], $f : D \to \mathbb{D}$ holomorphic, $\xi \in \partial D$ contact regular point, $v \in \mathbb{C}^d$, $v \neq 0$ then

$$d(z, \partial D)^{s_\xi(v)-1} \frac{\partial f}{\partial v}$$

is bounded on Koranyi regions. Moreover,

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$$\frac{\partial f}{\partial v} \xrightarrow{K'} c \neq 0.$$
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Normal component of the derivative

Proposition (Arosio, F., Gontard, Guerini '22)

We have

$$\langle \phi'(z), n_\xi \rangle$$

has nontangential limit $\phi'_N(1) > 0$.

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$\phi, \psi : \mathbb{D} \to D$ *complex geodesics with* $\phi(0) = \psi(0) = p$ *and* $\phi(1) = \psi(1) = \xi$, *then*

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The function Ω_ξ

Definition

It is well defined the function $\Omega_\xi : D \to (-\infty, 0)$ given by

$$\Omega_\xi(z) = -\frac{1}{\phi'_N(1)}$$

where $\phi : \mathbb{D} \to D$ a complex geodesic $\phi(0) = z$ and $\phi(1) = \xi$.

Main result

Theorem (Arosio-F. '23)

Let $D \subset \mathbb{C}^d$, $D' \subset \mathbb{C}^{d'}$ convex, $f : D \longrightarrow D'$ holomorphic, $p \in D$ and $p' \in D'$. Assume ∂D smooth and finite type in a neighborhood of $\xi \in \partial D$ and same for $\eta \in \partial D'$ (the K-limt of f at ξ). Then for all $v \in \mathbb{C}^d, u \in \mathbb{C}^{d'}$

$$\langle \frac{\partial f}{\partial v}, u \rangle d(z, \partial D)^{\frac{1}{L_\xi(v)} - \frac{1}{L_\eta(u)}}$$

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1. $\langle \frac{\partial f}{\partial n_\xi}, n_\eta \rangle \xrightarrow{K'} \frac{\Omega^D_\xi(p)}{\Omega^{D'}_\eta(p')} \lambda_{\xi,p,p'}$;
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- *Gromov hyperbolicity of convex finite type domains [Zimmer];*
- *Visibility and localisation of Kobayashi metric [Nikolov, Ökten, Thomas] and [Bracci, Nikolov, Thomas].*

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Interpretation of the Koranyi regions as tubular neighborhoods of real geodesics with endpoint ξ (Gromov hyperbolicity).

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Existence of horospheres and Julia Lemma [AFGG '22]

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THANK YOU FOR YOUR ATTENTION!