Estimates of invariant distances on strongly pseudoconvex domains

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This talk is going to be presented as follows:

1. Results
2. Corollaries
3. Methods

Results

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Theorem

Let Ω be a strongly pseudoconvex domain with $\mathcal{C}^{2,\alpha}$-boundary. Then, there exists constants $C > c > 0$ depending on Ω with

$$\log\left(1 + c\left(\frac{\|(z-w)_z\| + \|z-w\|^2 + \|z-w\|\sqrt{\delta_\Omega(z)}}{\sqrt{\delta_\Omega(z)\delta_\Omega(w)}}\right)\right) \leq k_\Omega(z,w)$$
$$\leq \log\left(1 + C\left(\frac{\|(z-w)_z\| + \|z-w\|^2 + \|z-w\|\sqrt{\delta_\Omega(z)}}{\sqrt{\delta_\Omega(z)\delta_\Omega(w)}}\right)\right), z, w \in \Omega.$$

Above $(z-w)_z$ is the complex normal component taken with respect to closest point in the boundary of Ω to z.

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Theorem

Let Ω be a strongly pseudoconvex domain with $\mathcal{C}^{3,1}$-boundary. Then, there exists constants $C > c > 0$ depending on Ω with

$$\log\left(1 + c\left(\frac{\|(z-w)_z\| + \|z-w\|^2 + \|z-w\|\sqrt{\delta_\Omega(z)}}{\sqrt{\delta_\Omega(z)\delta_\Omega(w)}}\right)\right) \leq b_\Omega(z,w) \leq$$

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Let Ω be a strongly pseudoconvex domain with $\mathcal{C}^{2,\alpha}$-boundary. Then, there exists a constant $c > 0$ depending on Ω such that

$$k_\Omega(z,w) \geq \log\left(1 + \frac{c\|(z-w)_z\|}{\sqrt{\delta_\Omega(z)\delta_\Omega(w)}}\right), \quad z, w \in \Omega.$$

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Note that a partial case of this result is proven by Kosinski-Nikolov and the full case is conjectured by them.

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Let Ω be a strongly pseudoconvex domain with $\mathcal{C}^{2,\alpha}$-boundary. Then, there exists a constant $c > 0$ such that if $z, w \in \Omega$ and $\gamma_{z,w} : I \to \Omega$ is a real Kobayashi geodesic joining z to w then we have that

$$D_{\Omega}^{\frac{1}{2}}(\gamma_{z,w}) \geq cl(\gamma_{z,w}) \tag{1}$$

where $D_{\Omega}(\gamma_{z,w}) := \max\{\delta_{\Omega}(\gamma(t)) : t \in I\}$ and $l(\gamma_{z,w})$ denotes the Euclidean length of $\gamma(I)$.

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Proposition:

Let Ω be a strongly-pseudoconvex domain with $\mathcal{C}^{2,\alpha}$-boundary. For any complex geodesic $\varphi : \Delta \to \Omega$ parametrized so that

$$D_\Omega(\varphi) := \max_{z \in \Delta} \delta_\Omega(\varphi(z)) = \delta_\Omega(\varphi(0))$$

we uniformly have, with constants depending on Ω that

$$D_\Omega^{\frac{1}{2}}(\varphi) \asymp d_e(\varphi(\Delta)) \asymp \max_{z \in \Delta} \|\varphi'(z)\| \tag{2}$$

Here $d_e(.)$ denotes the Euclidean diameter of a set.

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$$\frac{\|(z-w)_z\|}{\|z-w\|} + \sqrt{\delta_\Omega(z)} + \|z-w\| \asymp d_e(\varphi) \tag{3}$$

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Note that this result complements a result of Kosinski-Nikolov and also a result of Huang.

Corollaries

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Here,

$$g_\Omega(z,w) = 2\log\left(\frac{d^H_\Omega(\pi(z),\pi(w)) + \max\{\delta^{1/2}_\Omega(z), \delta^{1/2}_\Omega(w)\}}{(\delta_\Omega(z)\delta_\Omega(w))^{1/4}}\right)$$

where $d^H_\Omega(.,.)$ is the distance obtained by the Carnot-Carathéodory metric on $\partial\Omega$.

Let Ω be a strongly pseudoconvex domain with $\mathcal{C}^{2,\alpha}$ boundary and set $A_\Omega(z,w) = \|(z-w)_z\| + \|z-w\|^2 + \|z-w\|\sqrt{\delta_\Omega(z)}$. Our main result implies

$$k_\Omega(z,w) = \log\left(\frac{A_\Omega(z,w) + \sqrt{\delta_\Omega(z)\delta_\Omega(w)}}{\sqrt{\delta_\Omega(z)\delta_\Omega(x)}}\right) + \mathcal{O}(1).$$

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One may observe that $d_\Omega^H(\pi(z),\pi(w)) \asymp \sqrt{\|\pi(z)-\pi(w)\|^2 + \|(\pi(z)-\pi(w))_{\pi(z)}\|}$.

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One may observe that $d^H_\Omega(\pi(z),\pi(w)) \asymp \sqrt{\|\pi(z)-\pi(w)\|^2 + \|(\pi(z)-\pi(w))_{\pi(z)}\|}$. Using this fact we show that

$$A_\Omega(z,w) + \sqrt{\delta_\Omega(z)\delta_\Omega(w)} \asymp d^H_\Omega(\pi(z),\pi(w))^2 + \max\{\delta_\Omega(z),\delta_\Omega(w)\}$$

so on $\mathcal{C}^{2,\alpha}$-smooth case we get the estimate of Balogh-Bonk as a corollary.

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It is known by Pang that the Kobayashi-Royden metric is the "derivative" of the Lempert function. Observe that our estimate for the Lempert function yields the estimate given above for the Kobayashi-Royden metric. Similar reasoning allows us to reobtain similar estimates for the Bergman metric as well.

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$$d_e(\varphi(\Delta)) \asymp \sqrt{\delta_\Omega(\varphi(\zeta))} + \|(\varphi'(\zeta))_{\varphi(\zeta)}\| / \|(\varphi'(\zeta))\|.$$

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In particular, at $\partial\varphi$ we have

$$d_e(\varphi(\Delta)) \asymp \|(\varphi'(\zeta))_{\varphi(\zeta)}\| / \|(\varphi'(\zeta))\|.$$

Before we sketch our proofs let us note that if $\Omega = \mathbb{B}^n$ then

$$\log\left(1 + c\left(\frac{\|(z-w)_z\| + \|z-w\|\sqrt{\delta_\Omega(z)}}{\sqrt{\delta_\Omega(z)\delta_\Omega(w)}}\right)\right) \le k_\Omega(z,w)$$
$$\le \log\left(1 + C\left(\frac{\|(z-w)_z\| + \|z-w\|\sqrt{\delta_\Omega(z)}}{\sqrt{\delta_\Omega(z)\delta_\Omega(w)}}\right)\right), \quad z, w \in \Omega.$$

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On the other hand, if $\Omega = \{\mathrm{Re}(4z_1 - z_2^2) + |4z_2|^2 < 0\}$ is a pseudoconvex domain and for points $z = (\delta, 0)$, $w = (0, \epsilon)$ with δ/ϵ^2 tending to zero we have $\|(z-w)_z\| + \sqrt{\delta_\Omega(z)}\|z-w\|$ is $o(\|z-w\|^2)$.

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Proposition

Let p be a $\mathcal{C}^2$-smooth boundary point of a domain Ω in $\mathbb{C}^n$. Then the following conditions are equivalent:

1. Ω is strongly linearly convex at p;
2. $\liminf_{\Omega \ni z, w \to p} \frac{\|(z-w)_z\| + \|z-w\| \sqrt{\delta_\Omega(z)}}{\|z-w\|^2} > 0$;
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Hence, if Ω is strictly linearly convex with $\mathcal{C}^{2,\alpha}$-boundary then there exists $C > c > 0$ with

$$\log\left(1 + c\left(\frac{\|(z-w)_z\| + \|z-w\|\sqrt{\delta_\Omega(z)}}{\sqrt{\delta_\Omega(z)\delta_\Omega(w)}}\right)\right) \leq k_\Omega(z,w)$$
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Methods

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Lemma:

Let Ω be a strongly pseudoconvex domain with $\mathcal{C}^{2,\alpha}$-boundary.

1. Let $\epsilon > 0$ be small enough and $z, w \in \Omega$ be points satisfying $\|z - w\| \leq \epsilon$, $\delta_\Omega(z) \leq \epsilon$, $\delta_\Omega(w) \leq \epsilon$, $\frac{\|(z-w)_z\|}{\|z-w\|} \leq \epsilon$. Then z, w lies on the image of a complex geodesic $\varphi_{z,w}$.
2. For $\epsilon > 0$ small enough, if $z, w \in \Omega$ are points satisfying $\|z - w\| \leq \epsilon$, $\delta_\Omega(z) \leq \epsilon$, $\delta_\Omega(w) \leq \epsilon$, $\frac{(z-w)_z}{\|z-w\|} \leq \epsilon$ then there exists a unique real geodesic joining z to w. Furthermore, that real geodesic is induced from a complex geodesic.
3. We have a constant C depending on Ω such that for any complex geodesic $\varphi : \Delta \to \Omega$ with Euclidean diameter less than ϵ, we have $\frac{\|(x-y)_x\|}{\|x-y\|} \leq C\epsilon$ for any $x \neq y \in \varphi(\Delta)$.

Let $z_n, w_n \in \Omega$ be sequences tending to $p \in \partial\Omega$ with $\|(z_n - w_n)_{z_n}\|/\|z_n - w_n\|$ tending to zero. Then by the lemma above we have that they lie on complex geodesics $\varphi_n : \Delta \to \Omega$ whose diameters tend to zero. Let $x_n \in \varphi_n(\Delta)$ be the point that maximizes the boundary distance, set $p_n = \pi_\Omega(x_n)$.

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After a series of "local" biholomorphisms we have that $x_n = (1 - s_n + \gamma_n s_n^2, 0')$ $(|\gamma_n| \leq C \leq \infty)$, $p_n = p = (1, 0, ..., 0)$ and Ω near p is of the form

$$\Omega = \{1 - \|z\|^2 + \mathcal{O}(\|z - p\|^{2+\alpha}) < 0\}$$

so p is a strictly convex boundary point. In particular φ_n lies on a strictly convex neighbourhood of $\Omega \cap U_n$ of p. Thus it is a geodesic of $\Omega \cap U_n$. Here U_n's are uniformly close in $\mathcal{C}^{2,\alpha}$-topology.

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Note that these transformations verify $\|F(z) - F(w)\|_{F(z)} + \|F(z) - F(w)\|\sqrt{\delta_{F(\Omega)}(F(z))} + \|F(z) - F(w)\| \asymp \|z - w\| + \|z - w\|\sqrt{\delta_\Omega(z)} + \|z - w\|^2.$

Now we recall the automorphisms of the ball:

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As U_n are close in $\mathcal{C}^{2,\alpha}$-topology we can observe the following.

1. $A_t^{-1}(\Omega \cap U_n)$ tend to the ball in Hausdorff topology.
2. For any $\beta > -1$, $A_t^{-1}(\Omega \cap U_n) \cap \{\text{Re} z_1 > \beta\}$ converges to $\mathbb{B}^n$ in $\mathcal{C}^{2,\alpha}$-sense. (uniformly for each n)

Now, Lempert showed the following: Let Ω_n be a sequence of $\mathcal{C}^{2,\alpha}$-smooth strictly convex domains converging to a strictly convex domain Ω, then their complex geodesics converge to complex geodesics of Ω in $\mathcal{C}^{1,\alpha}$-sense.

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Let $z_n, w_n \in \varphi_n(\Delta)$ be arbitrary and $x_n \in \varphi_n(\Delta)$ with the maximal boundary distance point. Using the explicit form of $A_{t_n}^{-1}$ we observe the following:

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$$\|z_n - w_n\| \asymp \|\tilde{z}_n - \tilde{w}_n\| \sqrt{1 - t_n^2}$$

$$(1 - t_n^2)\delta_{\Omega_n}(\tilde{z}_n) \lesssim \delta_\Omega(z_n) \lesssim (1 - t_n^2)\delta_{\Omega_n}(\tilde{z}_n),$$

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These estimates allowed us to carry the behaviour of the real and complex geodesics of the ball to $\mathcal{C}^{2,\alpha}$-strongly pseudoconvex domains.

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Scaling has been a widely used tool, let us mention two of the technical differences in our proofs:

1. In order to show that $\tilde{\varphi}_n$ tends to $\psi(\zeta) = (0, \zeta, 0, ..., 0)$ in $\mathcal{C}^{1,\alpha}$-topology we needed to carefully chose the points with respect to which we scale with.
2. We were also able to observe the behaviour of $\|(z - w)_z\|$ in detail, after the local biholomorphisms in the construction and also the scaling.

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$$k_\Omega(\varphi(\zeta), \varphi(\omega)) = k_\Delta(\zeta, \omega) \geq \log\left(1 + \frac{c\|\zeta - \omega\|}{\delta_\Delta^{1/2}(\zeta)\delta_\Delta^{1/2}(\omega)}\right)$$

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$$\geq \log\left(1 + \frac{cD_\Omega(\varphi)\|\varphi(\zeta) - \varphi(\omega)\|}{(\max_{z\in\Delta}\|\varphi'(z)\|)\delta_\Omega^{1/2}(\varphi(\zeta))\delta_\Omega^{1/2}(\varphi(\omega))}\right).$$

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for any complex geodesic in Ω parametrized so that maximal boundary distance is attained at the origin.

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$$k_{\Omega}(\varphi(\zeta),\varphi(\omega)) \geq \log\left(1+\frac{c\|(\varphi(\zeta)-\varphi(\omega))_{\varphi(\zeta)}\|}{\delta_{\Omega}^{1/2}(\varphi(\zeta))\delta_{\Omega}^{1/2}(\varphi(\omega))}\right).$$

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$$k_{\Omega}(\varphi(\zeta),\varphi(\omega)) \geq \log\left(1+\frac{c\|(\varphi(\zeta)-\varphi(\omega))_{\varphi(\zeta)}\|}{\delta_{\Omega}^{1/2}(\varphi(\zeta))\delta_{\Omega}^{1/2}(\varphi(\omega))}\right).$$

Hence we proved

$$k_{\Omega}(z,w) \geq \log\left(1+\frac{c\|(z-w)_z\|}{\sqrt{\delta_{\Omega}(z)\delta_{\Omega}(w)}}\right), \quad z,w\in\Omega.$$

The lower bound follows by combining this lower bound with an estimate of Nikolov-Thomas.

Sketch of the upper bound in our main theorem: We recall the following upper bound by Nikolov-Andreev,

$$k_\Omega(z,w) \le \log\left(1 + \frac{C\|z-w\|}{\sqrt{\delta_\Omega(z)\delta_\Omega(w)}}\right), \quad z,w \in \Omega.$$

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$$k_\Omega(z,w) \leq \log\left(1 + \frac{C\|z-w\|}{\sqrt{\delta_\Omega(z)\delta_\Omega(w)}}\right), \quad z,w \in \Omega.$$

If z,w are "far" then $\|z-w\| \asymp \|z-w\|^2$. If z,w tend to an interior point $\|z-w\| \asymp \|z-w\|\sqrt{\delta_\Omega(z)}$. If z,w tend to a boundary point complex transversally then $\|z-w\| \asymp \|(z-w)_z\|$. Hence, in these cases the upper bound follows from the Nikolov-Andreev estimate.

We see that the upper bound can only fail for sequences $z_n, w_n \in \Omega$ be sequences tending to $p \in \partial\Omega$ with $\|(z_n - w_n)_{z_n}\| / \|z_n - w_n\|$ tending to zero. Let φ_n be the complex geodesic z_n, w_n is contained in.

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$$k_\Omega(z_n, w_n) \leq \log\left(1 + \frac{C\|z_n - w_n\| d_e(\varphi_n)}{\sqrt{\delta_\Omega(z_n)\delta_\Omega(w_n)}}\right).$$

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It remains to apply our estimate $\frac{\|(z-w)_z\|}{\|z-w\|} + \sqrt{\delta_\Omega(z)} + \|z - w\| \asymp d_e(\varphi)$.

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