Bounded geometry in several complex variables

Andrew Zimmer

July 13, 2023

University of Wisconsin-Madison

The Problem and the Plan

General Problem: Study domains in $\mathbb{C}^d$ with **weak boundary regularity** by assuming the **interior geometry** is sufficiently well behaved.

Interior geometry: Kähler-Einstein metric, Bergman metric, Kobayashi metric, ...

Goals of the talk:

- Convince you that bounded geometry assumptions can make difficult problems tractable
- State some questions and conjectures

The Problem and the Plan

Based on:

- "A lower bound for the Kähler-Einstein distance from the Diederich-Fornæss index" (PAMS 2021)
- "Two boundary rigidity results for holomorphic maps" (AJM 2022)
- "Compactness of the $\bar{\partial}$-Neumann problem on domains with bounded intrinsic geometry" (JFA 2021)

Outline:

- Notions of bounded geometry (slide 4)
- Three problems where bounded geometry helps:
  - Estimates on invariant distances (slide 10)
  - Boundary rigidity results (slide 14)
  - Compactness of the $\bar{\partial}$-Neumann operator (slide 22)
- Domains with bounded intrinsic geometry (slide 26)

Part 1: Notions of bounded geometry

Bounded geometry for Kähler manifolds

Definition: A complete Kähler manifold (X, g) has **bounded geometry** if

- g has positive injectivity radius
- for every integer $q \geq 0$, the curvature tensor R of g satisfies $\sup_X \|\nabla^q R\|_g < +\infty$.

Theorem [Wu-Yau 2020]: If (X, g) is a complete Kähler manifold, then the following are equivalent:

1. (X, g) has bounded geometry
2. there exist positive numbers $C > 1$ and $A_1, A_2, \ldots$ such that: for every $\zeta \in X$ there exists a holomorphic embedding $\Phi_\zeta : \mathbb{B} \to M$ where
   (a) $\Phi_\zeta(0) = \zeta$,
   (b) $C^{-1} g_{\mathrm{Euc}} \leq \Phi_\zeta^* g \leq C g_{\mathrm{Euc}}$,
   (c) for $q \geq 1$,

$$\left\| \Phi_\zeta^* g \right\|_{C^q(\mathbb{B})} \leq A_q.$$

Bounded geometry for Kähler manifolds

Theorem [Shi 1989]: If (X, g) is a complete Kähler manifold with bounded sectional curvature and positive injectivity radius, then for any $C > 1$ there exists a complete Kähler metric h on X such that

1. (X, h) has bounded geometry
2. $C^{-1}g \leq h \leq Cg$

Proof idea: Apply the Ricci flow to g for a short period of time.

Bounded geometry and pseudoconvex domains

Question: When does the Bergman metric have bounded geometry?

Partial Answer:

1. HHR/uniform squeezing domains (Liu-Sun-Yau 2004, Yeung 2009)
2. domains with bounded intrinsic geometry (Z. 2021)

Bounded geometry and pseudoconvex domains

Definition [Liu-Sun-Yau 2004, Yeung 2009]: A bounded domain $\Omega \subset \mathbb{C}^d$ is called a **holomorphic homogeneous regular domain** (HHR domain) if there exists $s \in (0,1)$ such that: for every $\zeta \in \Omega$ there exists a holomorphic embedding $\Phi_\zeta : \Omega \to \mathbb{C}^d$ with

$$\Phi_\zeta(\zeta) = 0 \text{ and } s\,\mathbb{B} \subset \Phi_\zeta(\Omega) \subset \mathbb{B}\,.$$

Theorem [Yeung 2009]: The Bergman metric on a HHR domain has bounded geometry. In particular, Ω is pseudoconvex.

Examples:

1. strongly pseudoconvex domains,
2. Kobayashi hyperbolic convex domains (e.g. bounded convex domains),
3. Kobayashi hyperbolic $\mathbb{C}$-convex domains,
4. homogeneous domains, and
5. the Teichmüller space of hyperbolic surfaces of genus g with n punctures.

Further, by definition, any domain biholomorphic to one of the domains listed is an HHR-domain.

Bounded geometry and pseudoconvex domains

Definition [Z. 2021]: A domain $\Omega \subset \mathbb{C}^d$ has **bounded intrinsic geometry** if there exists a complete Kähler metric g on Ω such that

(b.1) the metric g has bounded sectional curvature and positive injectivity radius,

(b.2) there exists a $\mathcal{C}^2$ function $\lambda : \Omega \to \mathbb{R}$ such that the Levi form of λ is uniformly bi-Lipschitz to g and $\|\partial\lambda\|_g$ is bounded on Ω.

Theorem [Z. 2021]: If $\Omega \subset \mathbb{C}^d$ is a domain with bounded intrinsic geometry, then:

1. The Bergman metric has bounded geometry. In particular, Ω is pseudoconvex.
2. The Bergman metric satisfies the definition of bounded intrinsic geometry.

Examples:

1. all known HHR-domains (known to me!),
2. finite type domains in $\mathbb{C}^2$,
3. simply connected domains which have a complete Kähler metric with pinched negative sectional curvature.

Part 2: Estimates on invariant distances

Conjecture (Folklore?): If

- Ω is a reasonable domain,
- $\delta_\Omega(z)$ is the Euclidean distance between z and $\partial\Omega$,
- dist_Ω is ether the Bergman, Kobayashi, or Kähler-Einstein distance on Ω, and
- $z_0 \in \Omega$,

then there exist $A, B > 0$ such that

$$\mathrm{dist}_\Omega(z, z_0) \geq -B + A \log \frac{1}{\delta_\Omega(z)}$$

for all $z \in \Omega$.

Estimates on invariant distances

Conjecture is known for

- strongly pseudoconvex domains,
- bounded convex domains (more generally bounded $\mathbb{C}$-convex domains),
- finite type domains in $\mathbb{C}^2$.

Other results:

- Błocki 2005: on a pseudoconvex domain with $\mathcal{C}^2$ boundary

$$\mathrm{dist}_{Berg}(z, z_0) \geq -A + B\frac{1}{\log\log\left(1/\delta_\Omega(z)\right)}\log\frac{1}{\delta_\Omega(z)}$$

- Mok-Yau 1980: on a pseudoconvex domain

$$\mathrm{dist}_{KE}(z, z_0) \geq -A + B\log\log\frac{1}{\delta_\Omega(z)}$$

Estimates on invariant distances

Theorem [Z. 2021]: If $\Omega \subset \mathbb{C}^d$ bounded pseudoconvex, $\partial\Omega$ is Lipschitz, and $z_0 \in \Omega$, then there exists $A, B > 0$ such that

$$\text{dist}_{KE}(z, z_0) \geq -A + B \log \frac{1}{\delta_\Omega(z)}.$$

Moreover, if the Bergman metric has bounded sectional curvature, then there exists $A, B > 0$ such that

$$\text{dist}_{Berg}(z, z_0) \geq -A + B \log \frac{1}{\delta_\Omega(z)}.$$

Estimates on invariant distances

Theorem [Z. 2021]: If $\Omega \subset \mathbb{C}^d$ bounded pseudoconvex, $\partial\Omega$ is Lipschitz, and $z_0 \in \Omega$, then there exists $A, B > 0$ such that

$$\mathrm{dist}_{KE}(z, z_0) \geq -A + B \log \frac{1}{\delta_\Omega(z)}.$$

Moreover, if the Bergman metric has <u>bounded sectional curvature</u>, then there exists $A, B > 0$ such that

$$\mathrm{dist}_{Berg}(z, z_0) \geq -A + B \log \frac{1}{\delta_\Omega(z)}.$$

Theorem [Z. 2020]: Suppose $\Omega \subset \mathbb{C}^d$ is a bounded pseudoconvex domain with positive hyperconvexity index $\alpha(\Omega) > 0$, g is a complete Kähler metric on Ω with $\mathrm{Ric}_g \geq -(2d-1)$, and dist_g is the distance associated to g. If $z_0 \in \Omega$ and $\epsilon > 0$, then there exists some $C = C(z_0, \epsilon) \geq 0$ such that

$$\mathrm{dist}_g(z, z_0) \geq -C + \left(\frac{\alpha(\Omega)}{2d-1} - \epsilon \right) \log \frac{1}{\delta_\Omega(z)}$$

Key idea: A Hopf-type lemma plus the Laplacian comparison theorem for Riemannian manifolds with Ricci curvature bounded below.

Part 3: Boundary rigidity results

Cartan's Uniqueness Theorem

Theorem [Cartan 1931]: If $\Omega \subset \mathbb{C}^d$ is a bounded domain, $f : \Omega \to \Omega$ is a holomorphic map, and there exists $z_0 \in \Omega$ such that

$$f(z) = z + \mathrm{O}\left(\|z - z_0\|^2\right),$$

then $f = \mathrm{id}$.

Question What happens when $z_0 \in \partial\Omega$?

Example:

- If $f : \mathbb{D} \to \mathbb{D}$ is a parabolic automorphism of the unit disc with fixed point $\xi_0 \in \partial\,\mathbb{D}$, then $f \neq \mathrm{id}$ and

$$f(z) = z + \mathrm{O}\left(\|z - \xi_0\|^2\right).$$

- (Burns-Krantz) If $f : \mathbb{D} \to \mathbb{D}$ is a holomorphic map and $f(z) = z + \mathrm{o}(\|z - \xi_0\|^3)$ for some $\xi_0 \in \partial\,\mathbb{D}$, then $f = \mathrm{id}$.

Boundary rigidity theorems for biholomorphisms

Theorem [Bell-Ligocka 1980, Baouendi-Ebenfelt-Rothschild 2000] Suppose $\Omega \subset \mathbb{C}^d$ is a bounded pseudoconvex domain with real analytic boundary and $\xi_0 \in \partial\Omega$. Then there exists $L = L(\xi_0) > 0$ such that: if $\varphi \in \mathrm{Aut}(\Omega)$ and

$$\varphi(z) = z + \mathrm{o}\left(\|z - \xi_0\|^L\right),$$

then $\varphi = \mathrm{id}$.

Steps in proof:

- Bell and Ligocka: φ extends to a CR-automorphism $\partial\Omega \to \partial\Omega$
- Baouendi-Ebenfelt-Rothschild: a finite jet determination theorem for CR-automorphisms

Theorem [Z. 2022] Suppose $\Omega \subset \mathbb{C}^d$ is bounded pseudoconvex domain where

(a) the Bergman metric g_Ω is complete and has bounded sectional curvature,

(b) $\sqrt{g_\Omega(X,X)} \lesssim \frac{1}{\delta_\Omega(z)} \|X\|$, and

(c) Ω satisfies an uniform interior cone condition (e.g. $\partial\Omega$ is Lipschitz),

then there exists $L > 0$ such that: if $\varphi \in \mathrm{Aut}(\Omega)$, $\xi_0 \in \partial\Omega$, and

$$\varphi(z) = z + \mathrm{o}\left(\|z - \xi_0\|^L\right),$$

then $\varphi = \mathrm{id}$.

Note: Conditions (a) and (b) are satisfied by HHR-domains or domains with bounded intrinsic geometry.

Boundary rigidity theorems for holomorphic self maps

Theorem [Burns-Krantz 1994]: Suppose $\Omega \subset \mathbb{C}^d$ is a bounded **strongly pseudoconvex** domain with C^6 boundary. If $f : \Omega \to \Omega$ is a holomorphic map and there exists $\xi_0 \in \partial\Omega$ such that

$$f(z) = z + \mathrm{o}\left(\|z - \xi_0\|^3\right),$$

then $f = \mathrm{id}$.

Question: What happens for general pseudoconvex domains?

Boundary rigidity theorems for holomorphic self maps

Conjecture [Burns-Krantz or maybe X. Huang (?)] Let $\Omega \subset \mathbb{C}^d$ be a pseudoconvex domain of **finite type** and suppose that $\xi_0 \in \partial\Omega$. Then there exists some m which depends on the geometry of $\partial\Omega$ at ξ_0 such that: if $f : \Omega \to \Omega$ is a holomorphic map and

$$f(z) = z + \mathrm{o}\left(\|z - \xi_0\|^m\right),$$

then $f = \mathrm{id}$.

Theorem [X. Huang 1995] Suppose that $\Omega \subset \mathbb{C}^d$ is a bounded **convex domain of finite type**. If $f : \Omega \to \Omega$ is a holomorphic map and there exists $\xi_0 \in \partial\Omega$ such that

$$f(z) = z + \mathrm{o}\left(\|z - \xi_0\|^m\right)$$

for some $m > 5$(line type at ξ_0), then $f = \mathrm{id}$.

Boundary rigidity theorems for holomorphic self maps

Theorem [Z. 2022] Suppose $\Omega \subset \mathbb{C}^d$ is a bounded convex domain with C^2 boundary. If $f : \Omega \to \Omega$ is a holomorphic map and there exists $\xi_0 \in \partial\Omega$ such that

$$f(z) = z + \mathrm{o}\left(\|z - \xi_0\|^4\right),$$

then $f = \mathrm{id}$.

Note This suggests that bounded geometry, not finite type is the driving force behind boundary rigidity results.

Conjecture Suppose $\Omega \subset \mathbb{C}^d$ is a **bounded convex domain** and $\xi_0 \in \partial\Omega$. Then there exists $L = L(\xi_0) > 0$ (depending on the regularity of $\partial\Omega$ at ξ_0) such that: if $f : \Omega \to \Omega$ is a holomorphic map and

$$f(z) = z + \mathrm{o}\left(\|z - \xi_0\|^L\right),$$

then $f = \mathrm{id}$.

Conjecture Suppose $\Omega \subset \mathbb{C}^d$ is a **bounded HHR domain** and $\xi_0 \in \partial\Omega$. Then there exists $L = L(s, \xi_0) > 0$ (depending on the regularity of $\partial\Omega$ at ξ_0 and the parameter s in the definition of HHR domains) such that: if $f : \Omega \to \Omega$ is a holomorphic map and

$$f(z) = z + \mathrm{o}\left(\|z - \xi_0\|^L\right),$$

then $f = \mathrm{id}$.

Part 4: Compactness of the $\bar{\partial}$-Neumann operator

Compactness of the $\bar{\partial}$-Neumann operator

Question: When is the $\bar{\partial}$-**Neumann operator** $N_q : L^2_{(0,q)}(D) \to L^2_{(0,q)}(D)$ compact?

Conjecture: q-dimensional analytic varieties in $\partial\Omega$ should be an obstruction to N_q being compact.

Theorem [Fu-Straube 1998]: If $\Omega \subset \mathbb{C}^d$ is a bounded **convex domain**, then the following are equivalent:

1. $N_q : L^2_{(0,q)}(\Omega) \to L^2_{(0,q)}(\Omega)$ is compact.
2. $\partial\Omega$ contains no q-dimensional analytic varieties.

Bounded intrinsic geometry - revisited

Definition: A domain $\Omega \subset \mathbb{C}^d$ has **bounded intrinsic geometry** if there exists a complete Kähler metric g on Ω such that

(b.1) the metric g has bounded sectional curvature and positive injectivity radius,

(b.2) there exists a $\mathcal{C}^2$ function $\lambda : \Omega \to \mathbb{R}$ such that the Levi form of λ is uniformly bi-Lipschitz to g and $\|\partial\lambda\|_g$ is bounded on Ω.

Observation: If $\Omega_1, \Omega_2 \subset \mathbb{C}^d$ are biholomorphic domains, then Ω_1 has bounded intrinsic geometry if and only if Ω_2 has bounded intrinsic geometry.

Examples:

1. all known HHR-domains (known to me!),
2. finite type domains in $\mathbb{C}^2$,
3. simply connected domains which have a complete Kähler metric with pinched negative sectional curvature.

Further, by definition, any domain biholomorphic to one of the domains listed above.

Characterizing compactness

Theorem [Z. 2021]: Suppose $\Omega \subset \mathbb{C}^d$ is a bounded domain with bounded intrinsic geometry and g_Ω is the Bergman metric on Ω. Then the following are equivalent:

1. $N_q : L^2_{(0,q)}(\Omega) \to L^2_{(0,q)}(\Omega)$ is compact
2. $\infty = \lim_{z \to \partial\Omega} \left(\text{the } q^{th} \text{ smallest singular value of } \left[g_{\Omega,z}\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_k}\right) \right] \right)$.

If, in addition, $\partial\Omega$ is $\mathcal{C}^0$, then the above conditions are equivalent to:

3. $\partial\Omega$ does not contain any q-dimensional analytic varieties.

Note:

- $\partial\Omega$ is $\mathcal{C}^0$ if for every $x \in \partial\Omega$ there exists a neighborhood U of x and there exists a linear change of coordinates which makes $U \cap \partial\Omega$ the graph of a continuous function.
- For convex domains (1) $\Leftrightarrow$ (3) is due to Fu-Straube (1998)

The proof: Uses results/ideas from complex geometry (Greene-Wu 1979, Wu-Yau 2020, ...) and several complex variables (Catlin, Fu-Straube, McNeal, ...)

Part 5: Domains with bounded intrinsic geometry

Bounded intrinsic geometry - Remarks on the definition I

Definition: A domain $\Omega \subset \mathbb{C}^d$ has **bounded intrinsic geometry** if there exists a complete Kähler metric g on Ω such that

(b.1) the metric g has bounded sectional curvature and positive injectivity radius,

(b.2) **there exists a $\mathcal{C}^2$ function $\lambda : \Omega \to \mathbb{R}$ such that the Levi form of λ is uniformly bi-Lipschitz to g and $\|\partial\lambda\|_g$ is bounded on Ω.**

The second condition is motivated by:

- Gromov's definition of Kähler hyperbolicity (1991)
- McNeal's definition of plurisubharmonic functions with self bounded complex gradient (2002)
- Vanishing results for L^2 cohomology (Donnelly-Fefferman 1983, Donnelly 1994, 1997, McNeal 2002)
- Results of Greene-Wu about simply connected negatively curved Kähler manifolds (1979)

Bounded intrinsic geometry - Remarks on the definition II

Definition: A domain $\Omega \subset \mathbb{C}^d$ has **bounded intrinsic geometry** if there exists a complete Kähler metric g on Ω such that

(b.1) the metric g has bounded sectional curvature and positive injectivity radius,

(b.2) there exists a $\mathcal{C}^2$ function $\lambda : \Omega \to \mathbb{R}$ such that **the Levi form of λ is uniformly bi-Lipschitz to g** and $\|\partial\lambda\|_g$ is bounded on Ω.

The Levi form of λ is

$$\mathcal{L}(\lambda) = \sum_{1 \leq \alpha, \beta \leq d} \frac{\partial\lambda}{\partial z_\alpha \partial \bar{z}_\beta} dz_\alpha \otimes d\bar{z}_\beta$$

and $\mathcal{L}(\lambda)$ is uniformly bi-Lipschitz to g if there exists $C > 1$ such that

$$\frac{1}{C} g \leq \mathcal{L}(\lambda) \leq Cg$$

Roughly, this means that g has a potential with bounded complex gradient. But it is helpful to allow $\mathcal{L}(\lambda)$ to be only <u>almost</u> equal to g.

Bounded intrinsic geometry - Remarks on the definition III

Definition: A domain $\Omega \subset \mathbb{C}^d$ has **bounded intrinsic geometry** if there exists a complete Kähler metric g on Ω such that

(b.1) the metric g has bounded sectional curvature and positive injectivity radius,

(b.2) there exists a $\mathcal{C}^2$ function $\lambda : \Omega \to \mathbb{R}$ such that the Levi form of λ is uniformly bi-Lipschitz to g and $\|\partial\lambda\|_g$ **is bounded on** Ω.

Recall,

$$\|\partial\lambda|_z\|_g = \max\left\{|\partial\lambda(X)| : X \in T_z^{(1,0)}\Omega,\ \|X\|_g \leq 1\right\}$$

for $z \in \Omega$.

The Bergman metric

Definition: A domain $\Omega \subset \mathbb{C}^d$ has **bounded intrinsic geometry** if there exists a complete Kähler metric g on Ω such that

(b.1) the metric g has bounded sectional curvature and positive injectivity radius,

(b.2) there exists a $\mathcal{C}^2$ function $\lambda : \Omega \to \mathbb{R}$ such that the Levi form of λ is uniformly bi-Lipschitz to g and $\|\partial\lambda\|_g$ is bounded on Ω.

Note: The metric g in the definition does not apriori have to be one of the "standard" Kähler metrics (like the Bergman or Kähler-Einstein metric), but....

Theorem [Z. 2021]: If $\Omega \subset \mathbb{C}^d$ is a domain, then the following are equivalent:

1. Ω has bounded intrinsic geometry,
2. the Bergman metric g_Ω satisfies the definition of bounded intrinsic geometry.

In particular, Ω is pseudoconvex.

The proof: uses results/ideas of Greene-Wu, Wu-Yau, ...

The Bergman kernel

Definition: A domain $\Omega \subset \mathbb{C}^d$ has **bounded intrinsic geometry** if there exists a complete Kähler metric g on Ω such that

(b.1) the metric g has bounded sectional curvature and positive injectivity radius,

(b.2) there exists a $\mathcal{C}^2$ function $\lambda : \Omega \to \mathbb{R}$ such that the Levi form of λ is uniformly bi-Lipschitz to g and $\|\partial\lambda\|_g$ is bounded on Ω.

Theorem [Z. 2021]: If $\Omega \subset \mathbb{C}^d$ is a domain, then the following are equivalent:

1. Ω has bounded intrinsic geometry,
2. the Bergman metric g_Ω satisfies the definition of bounded intrinsic geometry.

Warning: The above result says that W.O.L.O.G. we can assume that the metric g is the Bergman metric, but in general we cannot assume that λ is the standard potential for the Bergman metric:

Proposition [Z. 2021]: There exists a bounded domain $\Omega \subset \mathbb{C}^2$ biholomorphic to $\mathbb{D} \times \mathbb{D}$ with

$$\sup_{z \in \Omega} \|\partial \log \mathrm{B}_\Omega(z,z)\|_{g_\Omega} = \infty$$

where B_Ω is the Bergman kernel on Ω.

The Kobayashi metric

Recall, the **Kobayashi metric** on a domain $\Omega \subset \mathbb{C}^d$ is defined by

$$k_\Omega(z; v) = \inf \left\{ |\xi| : \xi \in \mathbb{C} \text{ and } \exists \varphi : \mathbb{D} \to \Omega \text{ holo. with } \varphi(0) = z,\ \varphi'(0)\xi = v \right\}$$

where $z \in \Omega$ and $v \in \mathbb{C}^d$.

Theorem [Z. 2021]: If $\Omega \subset \mathbb{C}^d$ is domain with bounded intrinsic geometry, g_Ω is the Bergman metric on Ω, and k_Ω is the Kobayashi metric on Ω, then there exists $C > 1$ such that

$$\frac{1}{C} k_\Omega(z; v) \leq \sqrt{g_{\Omega,z}(v,v)} \leq C k_\Omega(z; v)$$

for all $z \in \Omega$ and $v \in \mathbb{C}^d$.

Proof: Similar to estimates of Sibony (1979) for the Kobayashi metric

The pluricomplex Green function

The **pluricomplex Green function** $\mathrm{G}_\Omega(z,u) : \Omega \times \Omega \to \{-\infty\} \cup (-\infty, 0]$ is defined by

$$\mathrm{G}_\Omega(z,w) = \sup u(z)$$

where the supremum is taken over all negative plurisubharmonic functions u such that $u - \log \|z - w\|$ is bounded from above in a neighborhood of w.

Theorem [Z. 2021]: Suppose $\Omega \subset \mathbb{C}^d$ is a domain with bounded intrinsic geometry and G_Ω is the pluricomplex Green function on Ω. There exist $C, \tau > 0$ such that:

$$\log \mathrm{dist}_{Berg}(z,w) - C \leq \mathrm{G}_\Omega(z,w) \leq \log \mathrm{dist}_{Berg}(z,w) + C$$

for all $z, w \in \Omega$ with $\mathrm{dist}_{Berg}(z,w) \leq \tau$.

Local charts and plurisubharmonic functions with large Hessian

Lemma: If $\Omega \subset \mathbb{C}^d$ is a domain with bounded intrinsic geometry and g_Ω is the Bergman metric on Ω, then there exists $C > 1$ such that:

For every $\zeta \in \Omega$

1. there exists a holomorphic embedding $\Phi_\zeta : \mathbb{B} \to \Omega$ with $\Phi_\zeta(0) = \zeta$ and
$$C^{-1} g_{\mathrm{Euc}} \leq \Phi_\zeta^* g_\Omega \leq C g_{\mathrm{Euc}},$$
2. there exists a plurisubharmonic function $\phi_\zeta : \Omega \to [-1, 0]$ with
$$\mathcal{L}(\phi_\zeta) \geq C^{-1} g_\Omega \text{ on } \Phi_\zeta(\mathbb{B}).$$

The proof: Uses results of Shi (1989) and Wu-Yau (2020) concerning Kähler metrics with bounded geometry.

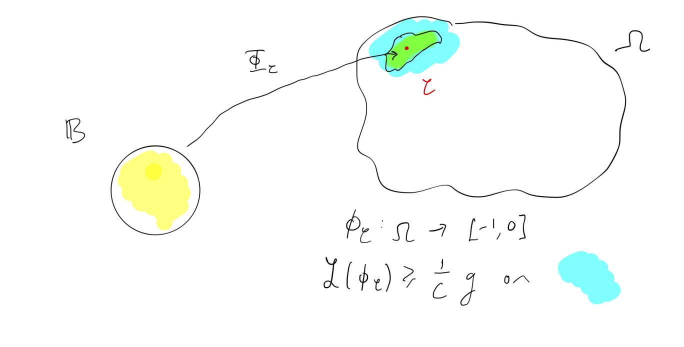
Motivated by: the "polydisk" constructions of Catlin for finite type domains in $\mathbb{C}^2$ (1989) and McNeal for finite type convex domains (1992, 1994).

Local charts and plurisubharmonic functions with large Hessian

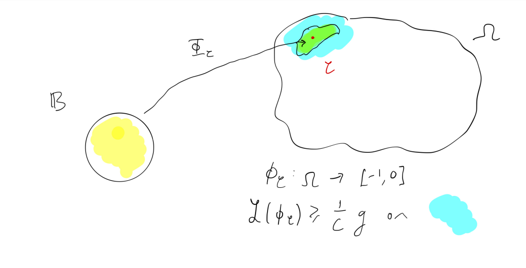
Lemma: If $\Omega \subset \mathbb{C}^d$ is a domain with bounded intrinsic geometry and g_Ω is the Bergman metric on Ω, then there exists $C > 1$ such that:

For every $\zeta \in \Omega$

1. there exists a holomorphic embedding $\Phi_\zeta : \mathbb{B} \to \Omega$ with $\Phi_\zeta(0) = \zeta$ and
$$C^{-1} g_{\mathrm{Euc}} \leq \Phi_\zeta^* g_\Omega \leq C g_{\mathrm{Euc}},$$
2. there exists a plurisubharmonic function $\phi_\zeta : \Omega \to [-1, 0]$ with
$$\mathcal{L}(\phi_\zeta) \geq C^{-1} g_\Omega \text{ on } \Phi_\zeta(\mathbb{B}).$$



Solving $\bar{\partial}$



The functions ϕ_ζ give us weights to use when solving $\bar{\partial}$. Which yields:

Lemma: For any $m \geq 0$, there exists $C_m > 0$ such that: if $\zeta \in \Omega$, $f : \Phi_\zeta(\mathbb{B}) \to \mathbb{C}$ is holomorphic, and $\int_{\Phi_\zeta(\mathbb{B})} |f|^2 \, dz < \infty$, then there exists a holomorphic function $F : \Omega \to \mathbb{C}$ where

$$\frac{\partial^{|\beta|} F}{\partial z^\beta}(\zeta) = \frac{\partial^{|\beta|} f}{\partial z^\beta}(\zeta)$$

for all multi-indices β with $|\beta| \leq m$ and

$$\int_\Omega |F|^2 \, dz \leq C_m \int_{\Phi_\zeta(\mathbb{B})} |f|^2 \, dz.$$

The Bergman kernel

Lemma: For any $m \geq 0$, there exists $C_m > 0$ such that: if $\zeta \in \Omega$, $f : \Phi_\zeta(\mathbb{B}) \to \mathbb{C}$ is holomorphic, and $\int_{\Phi_\zeta(\mathbb{B})} |f|^2 \, dz < \infty$, then there exists a holomorphic function $F : \Omega \to \mathbb{C}$ where

$$\frac{\partial^{|\beta|} F}{\partial z^\beta}(\zeta) = \frac{\partial^{|\beta|} f}{\partial z^\beta}(\zeta)$$

for all multi-indices β with $|\beta| \leq m$ and

$$\int_\Omega |F|^2 \, dz \leq C_m \int_{\Phi_\zeta(\mathbb{B})} |f|^2 \, dz.$$

This extension theorem allows us to estimate the Bergman kernel and metric (as in Greene-Wu 1979, Catlin 1989, McNeal 1994, ...)

Questions

Question: Suppose $\Omega \subset \mathbb{C}^d$ is a bounded domain. If the Bergman metric is complete and has bounded sectional curvature, then does the Bergman metric have positive injectivity radius? Do the derivatives of the curvature tensor have uniform bounds?

Question: What finite type domains have bounded intrinsic geometry? Note:

- finite type domains in $\mathbb{C}^2$ and convex finite type domains have bounded intrinsic geometry.
- there exist finite type domains in $\mathbb{C}^3$ which do not have bounded intrinsic geometry (Recall, the Kobayashi metric and Bergman metric are bi-Lipschitz on domains with bounded intrinsic geometry and this can fail for finite type domains in $\mathbb{C}^3$, see Diederich-Fornæss-Herbort 1984)

Question: Does every HHR-domain have bounded intrinsic geometry? Note: this would follow from a result claimed by Yeung, but his proof has a gap.

Question: If $\Omega \subset \mathbb{C}^d$ is bounded convex and $z_0 \in \Omega$, is $B_\Omega(\cdot, z_0)$ bounded?

Questions

Theorem [Burns-Krantz 1994]: Suppose $\Omega \subset \mathbb{C}^d$ is a bounded **strongly pseudoconvex** domain with C^6 boundary. If $f : \Omega \to \Omega$ is a holomorphic map and there exists $\xi_0 \in \partial\Omega$ such that

$$f(z) = z + \mathrm{o}\left(\|z - \xi_0\|^3\right),$$

then $f = \mathrm{id}$.

Question: Suppose $\Omega \subset \mathbb{C}^d$ is bounded pseudoconvex domain where

(a) Ω satisfies **some** bounded geometry condition,

(b) Ω satisfies an uniform interior cone condition (e.g. $\partial\Omega$ is Lipschitz),

then there exists $L > 0$ such that: if $f : \Omega \to \Omega$ is holomorphic, $\xi_0 \in \partial\Omega$, and

$$f(z) = z + \mathrm{o}\left(\|z - \xi_0\|^L\right),$$

then $f = \mathrm{id}$.

The End