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2-proper holomorphic images of classical Cartan domains

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based on a joint work with Gargi Ghosh

Dedicated to Peter Pflug on his 80th birthday

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To show their importance for the SCV, especially to the Lempert Theory, let us recall some of their properties.

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- The automorphism groups of $\mathbb{G}_2$ (respectively, $\mathbb{E}$) are known; they are isomorphic to that of $\mathbb{D}$ (respectively, $\mathbb{D}^2$).

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It turns out that in the case of bounded transitive domains we may assume that a is a linear isomorphism (in particular, $\mathcal{J}_\pi$ is the intersection of a hyperplane with D). Therefore, we lose no generality assuming that a is *a reflection*, i. e. a is a linear isomorphism of D and the set $\mathcal{J}_\pi$ is the intersection of a hyperplane with D.

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Cartan domains

We list all the Cartan domains below

- $\mathcal{R}_I(m \times n)$: The *classical Cartan domain of type I* consists of $m \times n$ complex matrices A such that the matrix $\mathbb{I}_m - AA^*$ is positive definite, (that is, $\mathbb{I}_m - AA^* > 0$ equivalently, $||A|| < 1$) where A^* denotes the adjoint of A and $\mathbb{I}_m$ denotes the identity matrix of order m.

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$$L_n := \left\{ z \in \mathbb{B}_n : \sqrt{\left(\sum_{j=1}^{n} |z_j|^2\right)^2 - \left|\sum_{j=1}^{n} z_j^2\right|^2} < 1 - \sum_{j=1}^{n} |z_j|^2 \right\}. \tag{3}$$

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Lie balls deliver additional examples of 2-proper holomorphic images

In fact, a natural 2-proper mapping holomorphic mapping on the Lie ball is the following

$$\Lambda_n : L_n \ni z \to (z_1^2, z_2, \ldots, z_n) \in \mathbb{L}_n. \tag{4}$$

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It turns out that $\mathbb{L}_2$ is biholomorphic to $\mathbb{G}_2$, $\mathbb{L}_3$ is biholomorphic to $\mathbb{E}$. As L_4 is isomorphic to $\mathcal{R}_I(2 \times 2)$ we also have a representation of the domain $\mathbb{L}_4$ in other way.

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By reduction to the two-dimensional case we get

Theorem

The domain $\mathbb{L}_n$ cannot be exhausted by domains biholomorphic to convex ones, $n \geq 2$.

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The domain $\mathbb{L}_n$ is not a Lu Qi-Keng domain for $n \geq 3$.

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Actually, as we shall see we may find all the automorphisms of $\mathbb{L}_n$.

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Proper holomorphic self-maps of $\mathbb{L}_n$ and the group $\mathrm{Aut}(\mathbb{L}_n)$

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It turns out that actualy we have no more automorphisms.

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$\mathrm{Aut}(L_{n-1}) \cong \mathrm{Aut}(\mathbb{L}_n)$ *for* $n \geq 3$.

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The Lempert Theorem for $\mathbb{L}_n$

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The domains $\mathbb{L}_n$ present a nice set of examples that could possibly be a good source for further research.

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- Another problem is to provide a geometric characterization of the domain $\mathbb{L}_n$ by the structure of its automorphism group (which is equal to $\mathrm{Aut}(L_{n-1})$) and some other geometric assumptions.