On the GL(n)-module structure of Lie nilpotent associative relatively free algebras

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Abstract

Let $K\langle X\rangle$ denote the free associative algebra generated by a set $X = \{x_1, \dots, x_n\}$ over a field K of characteristic 0. Let I_p, for any integer $p \geq 2$, denote the two-sided ideal in $K\langle X\rangle$ generated by all commutators of the form $[u_1, \dots, u_p]$ for $u_1, \dots, u_p \in K\langle X\rangle$. In this talk, we consider the quotient $K\langle X\rangle / I_{p+1}$ for all $p \geq 1$ and discuss its $\mathrm{GL}(n, K)$-module structure under the standard diagonal action. We start by describing some of the known cases for fixed small values of p. Then, for general p we give a bound on the values of partitions λ such that the irreducible $\mathrm{GL}(n, K)$-module V_λ appears in the decomposition of $K\langle X\rangle / I_{p+1}$ as a $\mathrm{GL}(n, K)$-module. As an application of our results, we take $K = \mathbb{C}$ and we consider the algebra of invariants $(\mathbb{C}\langle X\rangle / I_{p+1})^G$ for G being equal to one of the complex groups $\mathrm{SL}(n, \mathbb{C})$, $\mathrm{O}(n, \mathbb{C})$, $\mathrm{SO}(n, \mathbb{C})$, $\mathrm{Sp}(2s, \mathbb{C})$ (for $n = 2s$), or $\mathrm{UT}(n, \mathbb{C})$. By theorems of Domokos and Drensky, $(\mathbb{C}\langle X\rangle / I_{p+1})^G$ is finitely generated. In the talk, for each G as above we give an explicit upper bound on the degrees of homogeneous generators of $(\mathbb{C}\langle X\rangle / I_{p+1})^G$ in a minimal generating set. These results can be reformulated in the language of Classical Invariant Theory, so that for each G we obtain a criterion when a G-invariant of $\mathbb{C}\langle X\rangle$ belongs to I_p.

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