Product of spaces, dimension and universality

Abstract

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There are many papers, concerning the logarithmic law for the dimension of the product of spaces. In particular, examples of spaces, for which the logarithmic law is not true for its products, were constructed. We mension here some of them. First such spaces were given by L. Pontryagin. He construct two metric compact spases X and Y for which $dim(X) = dim(Y) = 2$ and $dim(X \times Y) = 3$. On the other hand V. Filippov (1971) construct two (non-metrizable) compact spaces X and Y for which $Ind(X) = ind(X) = 1$, $Ind(Y) = ind(Y) = 2$ and $Ind(X \times Y) \geq ind(X \times Y) > 3$. Later, A. Karassev and K. Kozlov (2015) (using a result of B. Pasynkov (1988)) proved that for these spaces we have $Ind(X \times Y) = ind(X \times Y) = 4$. Many examples of compact metric spaces, for which the logarithmic law is not true, are given by A. Dranishnikov (1988). He proved that for each natural numbers $n \leq m$ and each $r : n < r \leq m + n$, there are compact metric spaces X_n and X_m such that $dim(X_n) = n$, $dim(X_m) = m$ and $dim(X_n \times X_m) = r$.

The following propositions are corollaries of the given below main theorem.

Proposition 1. *For each separable metrizable space Y and each countable ordinals α and β in the non-empty class of all separable metrizable spaces X, for which $ind(X) = \alpha$ and $ind(Y \times X) = \beta$, there exists a universal element.*

Proposition 2. *For each completely regular space Y of weight $\leq \tau$ and each ordinals α and β in the non-empty class of all completely regular spaces X of weight $\leq \tau$, for which $ind(X) = \alpha$ and $ind(Y \times X) = \beta$, there exists a universal element.*

Proposition 3. *For each regular space Y of weight $\leq \tau$ and each ordinals α and β in the non-empty class of all regular spaces X of weight $\leq \tau$, for which $ind(X) = \alpha$ and $ind(Y \times X) = \beta$, there exists a universal element.*

Proposition 4. *For each* T_0*-space* Y *of weight* $\leq \tau$ *and each ordinals* α *and* β *in the non-empty class of all* T_0*-spaces* X *of weight* $\leq \tau$*, for which* $ind(X) = \alpha$ *and* $ind(Y \times X) = \beta$*, there exists a universal element.*

(For many Y, α and β, the considered in these propositions classes of spaces, may be empty.)

We note that although the Propositions $1 - 4$ have the similar formulations, they are independent each other.

The main result is the following theorem.

Теорема. *For each space* Y *of weight* $\leq \tau$*, each ordinals* α *and* β *and each saturated class* $\mathbb{S}$ *of spaces of weight* $\leq \tau$ *in the non-empty class of all elements* $X \in \mathbb{S}$*, for which* $ind(X) = \alpha$ *and* $ind(Y \times X) = \beta$*, there exists a universal element.*

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