

# A HJORTNAES-TYPE INTEGRAL FOR THE BASEL SERIES, AND SEVERAL OTHER RESULTS INVOLVING THE SPECIAL FUNCTIONS WHICH GENERALIZE IT

Lubomir Markov

April 1, 2023

At the Twelfth Scandinavian Mathematical Congress in 1953, Margrethe Munthe Hjortnaes presents (see [1]) an interesting transformation of the series  $\sum_{k=1}^{\infty} 1/k^3$  to a definite integral, namely the relation  $\zeta(3) = 10 \int_0^{\log[(1+\sqrt{5})/2]} t^2 \coth t dt$ . (Here and throughout,  $\zeta(z)$  denotes the Riemann zeta function.) In our talk we shall derive a similar representation for  $\zeta(2)$ . Recall that several special functions arise from the consideration of series of the form  $\sum_{k=1}^{\infty} \frac{c_k}{k^2} z^k$ , where  $c_k \in \{-1, 0, 1\}$ . Two of them are Ramanujan's inverse tangent integral  $T_2$  and Legendre's chi-function  $\chi_2$  (see [3], [4]), defined initially for  $|z| \leq 1$  by

$$T_2(z) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)^2}, \quad \chi_2(z) = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)^2},$$

and then extended by analytic continuation. We derive the power series expansion for the function  $z \mapsto T_2(\tan z)$  and use it to obtain several rapidly convergent numerical series involving zeta values, one example of which is the representation

$$\sum_{m=0}^{\infty} \frac{(2^{2m-1} - 1) \zeta(2m)}{2m+1} \frac{1}{4^{2m}} = \frac{G}{\pi},$$

where  $G$  denotes Catalan's constant. We also present a one-line solution to the Basel problem famously settled by Euler in 1734.

## REFERENCES

- [1] M. Munthe Hjortnaes, Overføring av rekken  $\sum_{k=1}^{\infty} (1/k^3)$  til et bestemt integral, *Tolfte Skandinaviske Matematikerkongressen, Lund, 1953* (1954), 211-213
- [2] L. Markov, A Functional Expansion and a New Set of Rapidly Convergent Series Involving Zeta Values, *Stud. Comput. Intelligence* **793** (2019), 267-276
- [3] S. Ramanujan, On the integral  $\int_0^x \frac{\tan^{-1} t}{t} dt$ , *J. Indian Math. Soc.* **7** (1915), 93-96
- [4] H.M. Srivastava and J. Choi, *Series Associated with the Zeta and Related Functions*, Kluwer Academic Publishers, Dordrecht, Boston and London, 2001

Lubomir Markov  
Department of Mathematics and CS  
Barry University  
11300 N.E. Second Avenue  
Miami Shores, FL 33161, USA  
[lmarkov@barry.edu](mailto:lmarkov@barry.edu)